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Gauge transformation of the third kind for $U(1)$-invariant coupled Schrödinger equations

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Abstract

We consider a large class of canonical coupled nonlinear Schrödinger equations invariant over the action of the $U(1)$-group. The most general nonlinearity is taken into account through a matrix that, without loss of generality, can be separated into the sum of a Hermitian matrix and an anti-Hermitian matrix. The $U(1)$-symmetry implies the existence of a set of continuity equations for the conserved densities, where the corresponding currents have, in general, a nonlinear structure. For this class of coupled Schrödinger equations we introduce a nonlinear gauge transformation which changes the nonlinear matrix into another one, purely Hermitian. Consequently, the currents are transformed in the standard bilinear form. Generalization to noncanonical systems is also discussed. Some examples are presented to illustrate the applicability of the method.

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1. Introduction and notations

In recent years, an increasing interest has been shown in systems of coupled nonlinear Schrödinger equations (CNSE), in particular after the invention of high-intensity lasers, which allowed [1] the experimental test of the pioneering theoretical works on solitons propagation in long-distance communications in optical fibres [2, 3]. In fact, single-mode optical fibres are not really single-mode, since different possible polarizations exist. A rigorous study of optical pulse propagation requires the use of CNSEs, in order to take into account the evolution of different polarized fields. In 1974 Manakov [4] introduced a CNSE starting from the cubic nonlinear Schrödinger equation (NSE), by considering the total field as a superposition of two, left and right polarized fields. When ultrashort pulses are transmitted through optical fibres, CNSEs with complex and derivative nonlinearities arise [5–10].

CNSEs are also employed in the study of other phenomenologies in physics, such as, in light propagation through a nonlinear birefringent medium, in nonrelativistic interactions in multi-species particle systems, in spinor Bose–Einstein condensation or in the description of micro-polar elastic solids [11–19].

Systems of CNSEs are more complicated to solve with respect to NSEs, and quite generally, the fruitful integrability techniques, developed for NSEs, fail when applied to systems of CNSEs. For this reason, it is of great interest to find methods able to reduce such systems to other more suitable forms.

By denoting with N_k the number of 'particles' of the kth species, in a multi-species system, many possible combinations of conserved multiplets can be realized. Two particular limiting cases are:

(a) All the quantities N_k are separately conserved. Such a situation is typical in nonrelativistic systems of multi-species interacting particles, where process of transmutation from a species to another one is forbidden.

(b) Only the quantity $N_{\rm tot} = \sum_k N_k$ is conserved. Relevant examples are given in the study of light propagation in optical fibres. Here, each species describes a polarization mode, and only the total intensity of the field is conserved.

Recently, in [20, 21] the authors studied a wide class of $U(1)$-invariant NSEs, containing a complex nonlinearity Λ. The current J, associated with the conserved density $\rho = |\psi|^2$ is, in general, nonlinear. Through the introducion of a unitary nonlinear transformation the complex nonlinearity Λ can be reduced in another one, which turns out to be purely real. As a consequence, the transformed current $\widetilde{J}$ assumes the standard bilinear form of the linear Schrödinger theory.

The same nonlinear gauge transformation has been generalized in [22] to the case of NSEs minimally coupled with an Abelian gauge field.

We recall that nonlinear transformations have been introduced previously in the literature to study different families of NSEs [23, 24] and CNSEs [25]. In [26], the term 'gauge transformations of third kind' was coined for the class of the unitary nonlinear transformations. Differently from the gauge transformations of first kind which have constant generators and those of the second kind which have generators depending on the space coordinate and eventually on time, the gauge transformations of third kind have generators depending functionally on the fields, often in a nonlinear manner.

On physical grounds they are named *gauge transformations* because, as stated by Feynmann and Hibbs [27], in a non-relativistic quantum mechanics, all measurements of observables are always accomplished through a measurement of position and time. Thus, quantum theories, for which the corresponding wavefunctions give the same probability density in space at all time, are in principle equivalent [26]. In particular, when the wavefunctions ψ and ϕ are related to each other by a unitary transformation, as in the gauge transformations which we are introducing, the two quantities $|\psi|^2 \equiv \rho_\psi = \rho_\phi \equiv |\phi|^2$, representing the density of probability of position for each time, are the same and, as a consequence, the fields ψ and ϕ describe the same physical system.

It is important to observe that nonlinear gauge transformations permit us to classify the Schrödinger equations in classes of equivalence. Any member belonging to the same class, in spite of its nonlinearity, describes the same physical system.

The purpose of the present work is to generalize the method previously introduced in [20–22] for the $U(1)$-invariant NSEs containing a complex nonlinearity to the case of $U(1)$-invariant CNSEs containing a non-Hermitian nonlinearity. The $U(1)$-invariance implies the existence of a set of continuity equations for the conserved densities ρ_k. The effect of the gauge transformation is to reduce this system of coupled equations to another one containing only a Hermitian nonlinearity. As a consequence, all the nonlinear currents associated with the continuity equations are reduced to the standard bilinear form. Preliminary results of the topics discussed in the following can be found in [28].

We introduce a wide class of CNSEs (in suitable units)

$$\mathrm{i}\Psi_t = -\widehat{A}\Psi_{xx} + \widehat{\Lambda}[\boldsymbol{\rho}, \boldsymbol{S}]\Psi + \widehat{V}(x)\Psi, \tag{1.1}$$

where $\Psi = (\psi_1, \ldots, \psi_p)$, $\boldsymbol{\rho} \equiv (\rho_1 \ldots, \rho_p)$ and $\boldsymbol{S} \equiv (S_1, \ldots, S_p)$ are p-dimensional vectors[1]. The components ψ_j of the vector Ψ are related to the real scalar fields ρ_j and S_j, components of the vectors $\boldsymbol{\rho}$ and $\boldsymbol{S}$, through the polar decomposition [29, 30]

$$\psi_j = \sqrt{\rho_j}\exp\left(\mathrm{i}S_j\right). \tag{1.2}$$

Hereinafter we denote the operator-valued matrix $\widehat{M}[\boldsymbol{v}]$ by a hat (the lower case letter $m[\boldsymbol{v}]$ denotes its entries) and use the notation between square brackets to indicate the functional dependence on the components of the vector $\boldsymbol{v} = (v_1, \ldots, v_p)$ and on its spatial derivatives of any order. Without loss of generality we assume the $p \times p$ matrix $\widehat{A}$ in a diagonal form. The potential $\widehat{V}(x)$ is a $p \times p$ diagonal matrix with real entries, describing the coupling between the vector field Ψ and an external force field.

We observe that any system of CNSEs can always be accommodated in the form given in equation (1.1) with a diagonal nonlinearity $\widehat{\Lambda}[\boldsymbol{\rho}, \boldsymbol{S}]$. Such nonlinearity can be separated in a Hermitian matrix $\widehat{W} = (\widehat{\Lambda} + \widehat{\Lambda}^\dagger)/2$ and an anti-Hermitian matrix $\mathrm{i}\widehat{\mathcal{W}} = (\widehat{\Lambda} - \widehat{\Lambda}^\dagger)/2$. Thus, without lost of generality, we can assume $\widehat{\Lambda}[\boldsymbol{\rho}, \boldsymbol{S}] = \widehat{W}[\boldsymbol{\rho}, \boldsymbol{S}] + \mathrm{i}\widehat{\mathcal{W}}[\boldsymbol{\rho}, \boldsymbol{S}]$, where the diagonal matrices $\widehat{W}[\boldsymbol{\rho}, \boldsymbol{S}]$ and $\widehat{\mathcal{W}}[\boldsymbol{\rho}, \boldsymbol{S}]$ have purely real entries. Such an assumption is only for convenience and does not imply any restriction on the form of the nonlinearity. Finally, we assume that all fields and their derivatives of any order vanish, at spatial infinity, in a sufficiently rapid way (uniform boundary conditions).

In the following we assume that the multi-component system (1.1) has q conserved multiplets of order p_k, with $k = 1, \ldots, q$ and $\sum_k p_k = p$, where $1 \leqslant q \leqslant p$.

The two particular cases (a) and (b) discussed previously are recognized for $q = p$ and $q = 1$, respectively.

Let us organize the fields ψ_i, belonging to the vector Ψ, in

$$\Psi \equiv \Big(\underbrace{\psi_{11}, \ldots, \psi_{1p_1}}_{\text{1st multiplet}}; \underbrace{\psi_{21}, \ldots, \psi_{2p_2}}_{\text{2nd multiplet}}; \ldots; \underbrace{\psi_{q1}, \ldots, \psi_{qp_q}}_{q\text{th multiplet}}\Big), \tag{1.3}$$

and, from now on, we relabel the fields ψ_i in ψ_{kl} where the first index k refers to the kth multiplet of order p_k, whereas the second index l, with $1 \leqslant l \leqslant p_k$, refers to the lth field inside the multiplet k.

The conservation of the q multiplets implies that equation (1.1) admits a set of q continuity equations

$$\rho_{k,t} + J_{k,x} = 0, \tag{1.4}$$

with

$$\rho_k = \sum_{l=1}^{p_k} |\psi_{kl}|^2. \tag{1.5}$$

Equation (1.4) assures the conservation of the quantities

$$N_k = \int_{-\infty}^{\infty} \rho_k \,\mathrm{d}x, \tag{1.6}$$

[1] Throughout this paper we indicate the derivatives with respect to x and t in $\psi_x \equiv \partial\psi/\partial x$ and $\psi_t \equiv \partial\psi/\partial t$.

where the integral is evaluated on the full real interval (uniform boundary conditions guarantee the convergence of the integral).

Once more we remark that, due to the non-Hermitian form of the nonlinearity $\widehat{\Lambda}[\boldsymbol{\rho}, \boldsymbol{S}]$, the currents J_k will have, in general, a nonlinear structure.

We introduce a nonlinear gauge transformation, $\Psi \to \Phi$, where $\Phi \equiv (\ldots, \phi_{kl}, \ldots)$ is a p-dimensional vector with components ϕ_{kl}, which transforms the system of CNSEs into another one, with a purely Hermitian nonlinearity. As a consequence, the transformed currents $\widetilde{J}_k$ assume the standard bilinear form of the coupled linear Schrödinger theory

$$\widetilde{J}_k = -\mathrm{i} \sum_{l=1}^{p_k} a_{kl} (\phi_{kl}^* \phi_{kl,x} - \phi_{kl,x}^* \phi_{kl}), \tag{1.7}$$

where a_{kl} are the entries of the matrix $\widehat{A}$ of equation (1.1).

The plan of the work is the following. In the next section we introduce the system under investigation whilst, in section 3, we study the restrictions imposed on the nonlinear potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ introduced in equation (2.1) by the set of continuity equations (1.4). The nonlinear gauge transformation is introduced in section 4 which contains our main results and, at the end of the same section, we present briefly the generalization of the method to the noncanonical systems. In section 5, we illustrate the applicability of the method with some explicit examples. Finally, section 6 is reserved for conclusions and discussions.

2. Coupled nonlinear Schrödinger equations

Let us consider a nonrelativistic canonical system described by the following Lagrangian density:

$$\mathcal{L}[\Psi^\dagger, \Psi] = \frac{\mathrm{i}}{2}\left(\Psi^\dagger \Psi_t - \Psi_t^\dagger \Psi\right) - \Psi_x^\dagger \widehat{A} \Psi_x - U[\boldsymbol{\rho}, \boldsymbol{S}] - \Psi^\dagger \widehat{V}(x) \Psi. \tag{2.1}$$

Because the theory is nonrelativistic, the Lagrangian contains only first-order time derivatives. The nonlinear potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ is a smooth real functional depending on the vector fields $\boldsymbol{\rho}$, $\boldsymbol{S}$ and their spatial derivatives of any order. Accounting for the uniform boundary conditions on the fields, the potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ vanishes, together with all its derivatives, at spatial infinity.

We introduce the action of the system

$$\mathcal{A} = \int_{\mathcal{R}} \mathcal{L}[\Psi^\dagger, \Psi] \, \mathrm{d}x \, \mathrm{d}t, \tag{2.2}$$

where the domain of integration is the whole real region $\mathcal{R} = \mathbb{R} \times \mathbb{R}$. The evolution equation for the vector field Ψ is given by the stationary trajectories of the action (2.2) and can be obtained from the following variational problem,

$$\delta \mathcal{A} = 0, \tag{2.3}$$

where the variation in equation (2.3) is performed with respect to the $2p$-dimensional vector $\Omega \equiv (\Psi^\dagger, \Psi)$.

From equation (2.3) we obtain

$$\mathrm{i}\Psi_t = -\widehat{A}\Psi_{xx} + \frac{\delta}{\delta \Psi^\dagger} \int_{\mathcal{R}} U[\boldsymbol{\rho}, \boldsymbol{S}] \, \mathrm{d}x \, \mathrm{d}t + \widehat{V}(x)\Psi, \tag{2.4}$$

and its Hermitian conjugate, which form a system of $2p$ nonlinear coupled Schröedinger equations.

Taking into account of the polar decomposition of the fields ψ_{kl}, in the real scalar fields ρ_{kl} and S_{kl}, given by

$$\psi_{kl} = \sqrt{\rho_{kl}}\exp\left(\mathrm{i}S_{kl}\right), \tag{2.5}$$

and their inverse formulae

$$\rho_{kl} = |\psi_{kl}|^2, \tag{2.6}$$

$$S_{kl} = \frac{\mathrm{i}}{2}\log\left(\frac{\psi_{kl}{}^*}{\psi_{kl}}\right), \tag{2.7}$$

we can express the variation $\delta/\delta\psi_{kl}^*$ as

$$\frac{\delta}{\delta\psi_{kl}^*} = \psi_{kl}\left(\frac{\delta}{\delta\rho_{kl}} + \frac{\mathrm{i}}{2\rho_{kl}}\frac{\delta}{\delta S_{kl}}\right). \tag{2.8}$$

Then, each component of equation (2.4) can be written in

$$\begin{aligned}
\mathrm{i}\psi_{kl,t} &= -a_{kl}\psi_{kl,xx} + \frac{\delta}{\delta\psi_{kl}^*}\int U[\boldsymbol{\rho},\boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t + v_{kl}(x)\psi_{kl} \\
&= -a_{kl}\psi_{kl,xx} + \left(\frac{\delta}{\delta\rho_{kl}} + \frac{\mathrm{i}}{2\rho_{kl}}\frac{\delta}{\delta S_{kl}}\right)\left(\int U[\boldsymbol{\rho},\boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t\right)\psi_{kl} + v_{kl}(x)\psi_{kl} \\
&= -a_{kl}\psi_{kl,xx} + \frac{\delta}{\delta\rho_{kl}}\left(\int U[\boldsymbol{\rho},\boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t\right)\psi_{kl} \\
&\quad + \frac{\mathrm{i}}{2\rho_{kl}}\frac{\delta}{\delta S_{kl}}\left(\int U[\boldsymbol{\rho},\boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t\right)\psi_{kl} + v_{kl}(x)\psi_{kl},
\end{aligned} \tag{2.9}$$

and can be posed in the following matrix form,

$$\mathrm{i}\Psi_t = -\widehat{A}\Psi_{xx} + \left(\widehat{W}[\boldsymbol{\rho},\boldsymbol{S}] + \mathrm{i}\widehat{\mathcal{W}}[\boldsymbol{\rho},\boldsymbol{S}]\right)\Psi + \widehat{V}(x)\Psi, \tag{2.10}$$

where

$$\widehat{W}[\boldsymbol{\rho},\boldsymbol{S}] = \mathrm{diag}\left(\frac{\delta}{\delta\rho_{kl}}\int U[\boldsymbol{\rho},\boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t\right), \tag{2.11}$$

$$\widehat{\mathcal{W}}[\boldsymbol{\rho},\boldsymbol{S}] = \mathrm{diag}\left(\frac{1}{2\rho_{kl}}\frac{\delta}{\delta S_{kl}}\int U[\boldsymbol{\rho},\boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t\right). \tag{2.12}$$

Finally, by using the polar decomposition (2.5), equation (2.10) can be separated in a system of $2p$ nonlinear real coupled equations

$$\rho_{kl,t} + 2a_{kl}(\rho_{kl}S_{kl,x})_x - 2\rho_{kl}\mathcal{W}_{kl}[\boldsymbol{\rho},\boldsymbol{S}] = 0, \tag{2.13}$$

$$S_{kl,t} + a_{kl}(S_{kl,x})^2 - a_{kl}\frac{(\sqrt{\rho_{kl}})_{xx}}{\sqrt{\rho_{kl}}} + w_{kl}[\boldsymbol{\rho},\boldsymbol{S}] + v_{kl}(x) = 0. \tag{2.14}$$

The first set of equations (2.13) describes the time evolution of the fields ρ_{kl}, whilst the second set of equations (2.14) is a system of p-coupled Hamilton–Jacobi-like equations for the fields S_{kl}.

3. $U(1)$-symmetry

In the following we consider only those systems written in the form (2.10) admitting the set of q continuity equations (1.4). This imposes some restrictions on the functional dependence of the potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ with respect to the fields $\boldsymbol{\rho}$ and $\boldsymbol{S}$ that we derive in this section.

Let us begin by recalling the relation

$$\frac{\delta}{\delta S_{kl}} = \frac{\partial}{\partial S_{kl}} - \frac{\partial}{\partial x}\frac{\delta}{\delta S_{kl,x}}, \tag{3.1}$$

which follows directly from the definition of functional derivative [31]. By taking into account the expression of the matrix $\widehat{W}$, given in equation (2.12), equation (2.13) can be written in

$$\rho_{kl,t} + \left(2a_{kl}\rho_{kl}S_{kl,x} + \frac{\delta}{\delta S_{kl,x}}\int U\left[\boldsymbol{\rho}, \boldsymbol{S}\right]\,\mathrm{d}x\,\mathrm{d}t\right)_x - \frac{\partial}{\partial S_{kl}}\int U\left[\boldsymbol{\rho}, \boldsymbol{S}\right]\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.2}$$

By summing on the index l, with $1 \leqslant l \leqslant p_k$, we obtain

$$\rho_{k,t} + (j_k + \mathcal{J}_k[\boldsymbol{\rho}, \boldsymbol{S}])_x + I_k[\boldsymbol{\rho}, \boldsymbol{S}] = 0, \tag{3.3}$$

where we have defined

$$\rho_k = \sum_{l=1}^{p_k}\rho_{kl}, \tag{3.4}$$

which is the total density of the kth multiplet and

$$j_k = \sum_{l=1}^{p_k} j_{kl}, \tag{3.5}$$

which is the total current of particle of the kth multiplet, with

$$j_{kl} = 2a_{kl}\rho_{kl}S_{kl,x}. \tag{3.6}$$

In equation (3.3) we have posed

$$\mathcal{J}_k[\boldsymbol{\rho}, \boldsymbol{S}] = \sum_{l=1}^{p_k}\mathcal{J}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}], \tag{3.7}$$

with

$$\mathcal{J}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] = \frac{\delta}{\delta S_{kl,x}}\int U[\boldsymbol{\rho}, \boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t, \tag{3.8}$$

and

$$I_k[\boldsymbol{\rho}, \boldsymbol{S}] = \sum_{l=1}^{p_k} I_{kl}[\boldsymbol{\rho}, \boldsymbol{S}], \tag{3.9}$$

with

$$I_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] = -\frac{\partial}{\partial S_{kl}}\int U[\boldsymbol{\rho}, \boldsymbol{S}]\,\mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

Within these notations the matrix $\widehat{\mathcal{W}}$ assumes the expression

$$\widehat{\mathcal{W}}[\boldsymbol{\rho}, \boldsymbol{S}] = \mathrm{diag}\left[-\frac{1}{2\rho_{kl}}\left(I_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] + \mathcal{J}_{kl,x}[\boldsymbol{\rho}, \boldsymbol{S}]\right)\right]. \tag{3.11}$$

Because we are interested in $U(1)$-invariant systems conserving the quantities N_k, introduced in equation (1.6), we require that equations (3.3) become a set of q continuity equations for the densities ρ_k. This implies that the functionals $I_k[\boldsymbol{\rho}, \boldsymbol{S}]$ can be expressed as the derivatives of a set of functionals $G_k[\boldsymbol{\rho}, \boldsymbol{S}]$:

$$I_k[\boldsymbol{\rho}, \boldsymbol{S}] = G_{k,x}[\boldsymbol{\rho}, \boldsymbol{S}], \tag{3.12}$$

where the x-derivative of $G_k[\boldsymbol{\rho}, \boldsymbol{S}]$ is given by

$$G_{k,x}[\boldsymbol{\rho}, \boldsymbol{S}] = \frac{\delta}{\delta \rho_{ij}} G_k[\boldsymbol{\rho}, \boldsymbol{S}] \rho_{ij,x} + \frac{\delta}{\delta S_{ij}} G_k[\boldsymbol{\rho}, \boldsymbol{S}] S_{ij,x}, \tag{3.13}$$

and a sum on the repeated indices i and j is assumed.

We remark that the expression of the functionals $G_k[\boldsymbol{\rho}, \boldsymbol{S}]$ is determined univocally from the nonlinear potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ through equations (3.9), (3.10) and (3.12). Thus, conditions (3.12) select the class of the Lagrangians (2.1) in which the method that we are introducing in the next section can be performed.

If conditions (3.12) are accomplished, equations (3.3) form a system of q continuity equations, like in (1.4), where the nonlinear currents J_k are given by

$$J_k = j_k + \mathcal{J}_k[\boldsymbol{\rho}, \boldsymbol{S}] + G_k[\boldsymbol{\rho}, \boldsymbol{S}]. \tag{3.14}$$

It is worth noting that the evolution equations for the single densities ρ_{kl} actually are given by

$$\rho_{kl,t} + (j_{kl} + \mathcal{J}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}])_x + I_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] = 0. \tag{3.15}$$

The quantities I_{kl} are responsible for the non-conservation of the single components ρ_{kl}. They take into account the transmutation of the component ρ_{kl} in the other components belonging to the same multiplet k.

Let us now inquire on the conditions imposed on the nonlinear potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ by equations (3.12). We recall that, as follows from the Noether theorem, equations (1.4) are a consequence of the invariance of the Lagrangian (2.1) with respect to a global unitary transformation

$$\Psi \to \Phi = \widehat{U}\Psi, \tag{3.16}$$

with

$$\widehat{U} = \mathrm{diag}[\exp(\mathrm{i}\boldsymbol{\epsilon})], \tag{3.17}$$

where

$$\boldsymbol{\epsilon} \equiv \big(\underbrace{\epsilon_1, \ldots, \epsilon_1}_{p_1 \text{ times}}; \underbrace{\epsilon_2, \ldots, \epsilon_2}_{p_2 \text{ times}}; \ldots; \underbrace{\epsilon_q, \ldots, \epsilon_q}_{p_q \text{ times}}\big) \tag{3.18}$$

are the constant parameters of the transformation.

Equation (3.16) shifts the phase $\boldsymbol{S}$ of the field Ψ according to the relation $\boldsymbol{S} \to \boldsymbol{\Sigma} = \boldsymbol{S} + \boldsymbol{\epsilon}$, where $\boldsymbol{\Sigma}$ is the phase of the transformed field Φ.

As is known, the Lagrangian (2.1) is invariant under the transformation (3.16) if the nonlinear potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ changes as

$$\delta U[\boldsymbol{\rho}, \boldsymbol{S}] = -\sum_{k=1}^{q} \epsilon_k G_{k,x}[\boldsymbol{\rho}, \boldsymbol{S}], \tag{3.19}$$

where $G_k[\boldsymbol{\rho}, \boldsymbol{S}]$ are arbitrary functionals. We recall that in this way the motion equation (2.4) does not change because the Lagrangian density (2.1) is always defined modulo a total derivative of an arbitrary functional (null Lagrangian). Accounting for the independence of the parameters ϵ_k, from equation (3.19) it follows that

$$\sum_{l=1}^{p_k} \frac{\partial}{\partial S_{kl}} \int U[\boldsymbol{\rho}, \boldsymbol{S}]\, \mathrm{d}x\, \mathrm{d}t = -G_{k,x}[\boldsymbol{\rho}, \boldsymbol{S}], \tag{3.20}$$

which coincides with equation (3.12).

In addition, because the parameters ϵ_k are constants, the potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ can depend on the phases S_{kl} through their spatial derivatives of any order.

In particular, when $p = q$ with $p_k = 1$, for $k = 1, \ldots, p$, the system conserves separately each component ρ_{k1} and, from equation (3.15), it follows that all the quantities I_{k1} must vanish. In this case the invariance of the Lagrangian under transformation (3.16) requires that the potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ depends only on the spatial derivatives of the fields S_{k1}, in accordance with the results obtained in [22].

In this case, the matrix $\widehat{\mathcal{W}}$ assumes the more simple expression

$$\widehat{\mathcal{W}}[\boldsymbol{\rho}, \boldsymbol{S}] = \mathrm{diag}\left(-\frac{\mathcal{J}_{k1,x}[\boldsymbol{\rho}, \boldsymbol{S}]}{2\rho_{k1}}\right), \tag{3.21}$$

where the functionals $\mathcal{J}_{k1}[\boldsymbol{\rho}, \boldsymbol{S}]$ are defined in equation (3.8), by posing $p_k = 1$.

4. Gauge transformation of the third kind

Let us introduce the following nonlinear transformation,

$$\Psi(x,t) \to \Phi(x,t) = \widehat{\mathcal{U}}[\boldsymbol{\rho}, \boldsymbol{S}]\Psi(x,t), \tag{4.1}$$

where $\widehat{\mathcal{U}}$ is a diagonal and unitary matrix: $\widehat{\mathcal{U}}^\dagger = \widehat{\mathcal{U}}^{-1}$.

The purpose of transformation (4.1) is to change the CNSE (2.10) into another one containing only a purely Hermitian nonlinearity $\widehat{W}' = (\widehat{W}')^\dagger$.

As a consequence, the nonlinear currents J_k, given in equation (3.14), are transformed into $J_k \to \widetilde{J}_k$ where

$$\widetilde{J}_k = \sum_{l=1}^{p_k} \widetilde{j}_{kl}, \tag{4.2}$$

with

$$\widetilde{j}_{kl} = 2a_{kl}\rho_{kl}\Sigma_{kl,x}, \tag{4.3}$$

and Σ_{kl} are the phases of the new fields ϕ_{kl}.

Since the matrix $\widehat{\mathcal{U}}$ is diagonal and unitary, we still have

$$\rho_{kl} = |\psi_{kl}|^2 = |\phi_{kl}|^2, \tag{4.4}$$

whilst the phases Σ_{kl} are related to ϕ_{kl} through the relation

$$\Sigma_{kl} = \frac{\mathrm{i}}{2} \ln\left(\frac{\phi_{kl}^*}{\phi_{kl}}\right). \tag{4.5}$$

Without lost of generality, the matrix $\widehat{\mathcal{U}}$ can be written as

$$\widehat{\mathcal{U}}[\boldsymbol{\rho}, \boldsymbol{S}] = \mathrm{diag}\,[\exp(\mathrm{i}\boldsymbol{\sigma}[\boldsymbol{\rho}, \boldsymbol{S}])], \tag{4.6}$$

where $\boldsymbol{\sigma} \equiv (\ldots, \sigma_{kl}, \ldots)$ is a p-dimensional vector, containing the generators $\sigma_{kl}[\boldsymbol{\rho}, \boldsymbol{S}]$ of the transformation (4.1), which are real functionals. They relate, through equation (4.1), the phase $\boldsymbol{\Sigma}$ of the new field Φ with the phase $\boldsymbol{S}$ of the old field Ψ, according to the relation

$$\boldsymbol{\Sigma} = \boldsymbol{S} + \boldsymbol{\sigma}[\boldsymbol{\rho}, \boldsymbol{S}]. \tag{4.7}$$

When equation (4.7) is invertible, we can express the phases S_{kl} as functionals of the fields $\boldsymbol{\rho}$ and $\boldsymbol{\Sigma}$.

Let us write the generators $\sigma_{kl}[\boldsymbol{\rho}, \boldsymbol{S}]$ as

$$\sigma_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] = \frac{1}{2a_{kl}} \int \frac{1}{\rho_{kl}} (\mathcal{J}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] + \mathcal{R}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}])\, \mathrm{d}x, \tag{4.8}$$

where $\mathcal{R}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}]$ are arbitrary real functionals related to $G_k[\boldsymbol{\rho}, \boldsymbol{S}]$, introduced in equation (3.12), through the relations

$$\sum_{l=1}^{p_k} \mathcal{R}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] = G_k[\boldsymbol{\rho}, \boldsymbol{S}]. \tag{4.9}$$

Actually, accounting for equations (4.8), it follows that equation (4.1) defines a wide class of transformations, one for every choice of the set of functionals $\mathcal{R}_{kl}$. Each of these nonlinear gauge transformations changes the initial system (2.10), with the nonlinearity $\widehat{W}[\boldsymbol{\rho}, \boldsymbol{S}] + \mathrm{i}\widehat{\mathcal{W}}[\boldsymbol{\rho}, \boldsymbol{S}]$, in another one with a purely Hermitian matrix $\widehat{W}'[\boldsymbol{\rho}, \boldsymbol{S}]$.

In fact, by performing the transformation (4.1), equation (2.10) becomes

$$\mathrm{i}\Phi_t = -\widehat{A}\Phi_{xx} + (\widehat{W}_0[\boldsymbol{\rho}, \boldsymbol{\Sigma}] + \mathrm{i}\widehat{\mathcal{W}}_0[\boldsymbol{\rho}, \boldsymbol{\Sigma}])\Phi + \widehat{V}(x)\Phi, \tag{4.10}$$

with

$$\widehat{W}_0 = \mathrm{diag}\,[w_{kl} + a_{kl}(\sigma_{kl,x})^2 - 2a_{kl}\Sigma_{kl,x}\sigma_{kl,x} - \sigma_{kl,t}], \tag{4.11}$$

and

$$\widehat{\mathcal{W}}_0 = \mathrm{diag}\left(\frac{\mathcal{F}_{kl}}{\rho_{kl}}\right), \tag{4.12}$$

where the functionals $\mathcal{F}_{kl}$, given by

$$\mathcal{F}_{kl} = I_{kl} - \mathcal{R}_{kl,x}, \tag{4.13}$$

with I_{kl} introduced in equation (3.10), fulfil the relations

$$\sum_{l=1}^{p_k} \mathcal{F}_{kl} = 0, \tag{4.14}$$

as can be verified by employing equations (3.9), (3.12) and (4.9).

It is easy to see that, as a consequence of equations (4.14), equation (4.10) admits the following set of continuity equations,

$$\rho_{k,t} + \widetilde{J}_{k,x} = 0, \tag{4.15}$$

where the currents $\widetilde{J}_k$, given in equations (4.2) and (4.3), have the standard form of the linear quantum mechanics.

This last result implies that the matrix $\widehat{W}_0[\boldsymbol{\rho}, \boldsymbol{\Sigma}] + \mathrm{i}\widehat{\mathcal{W}}_0[\boldsymbol{\rho}, \boldsymbol{\Sigma}]$ can be rearranged in a purely Hermitian matrix. In fact, equation (4.10) can be rewritten in

$$\mathrm{i}\Phi_t = -\widehat{A}\Phi_{xx} + \widehat{W}'[\boldsymbol{\rho}, \boldsymbol{\Sigma}]\Phi + \widehat{V}(x)\Phi, \tag{4.16}$$

where the matrix $\widehat{W}'[\boldsymbol{\rho}, \boldsymbol{\Sigma}]$ assumes the following block form,

$$\widehat{W}'[\boldsymbol{\rho}, \boldsymbol{\Sigma}] = \mathrm{diag}\,(\widehat{W}'_k[\boldsymbol{\rho}, \boldsymbol{\Sigma}]), \tag{4.17}$$

being the $p_k \times p_k$ matrices $\widehat{W}'_k[\boldsymbol{\rho}, \boldsymbol{\Sigma}] = \widehat{D}_k[\boldsymbol{\rho}, \boldsymbol{\Sigma}] + \widehat{C}_k[\boldsymbol{\rho}, \boldsymbol{\Sigma}]$, composed by a diagonal part

$$\widehat{D}_k = \mathrm{diag}\,[w_{kl} + a_{kl}(\sigma_{kl,x})^2 - 2a_{kl}\Sigma_{kl,x}\sigma_{kl,x} - \sigma_{kl,t}], \tag{4.18}$$

with purely real entries, and an off-diagonal part

$$(\widehat{C}_k)_{lm} = \mathrm{i}\frac{\mathcal{F}_{kl} - \mathcal{F}_{km}}{2p_k\sqrt{\rho_{kl}\rho_{km}}}\,\mathrm{e}^{\mathrm{i}(S_{kl}-S_{km})}, \tag{4.19}$$

which result to be Hermitian matrices: $\widehat{C}_k = \widehat{C}_k^\dagger$. This is our main result.

In the case $p = q$ the functionals $\mathcal{F}_{k1}$ vanish and the matrix $\widehat{W}'$ is reduced to a diagonal form so that equation (4.16) contains now only a purely real nonlinearity given by

$$\widehat{W}' = \text{diag}\,[w_k + a_k(\sigma_{k,x})^2 - 2a_k\,\Sigma_{k,x}\sigma_{k,x} - \sigma_{k,t}], \tag{4.20}$$

which is in accordance with the results presented in [28].

We observe that because the Lagrangian (2.1) is $U(1)$ invariant, the arbitrary integration constant, deriving from the definition (4.8), does not produce any effect and can be posed equal to zero. Moreover, the last term $\sigma_{kl,t}$ in equation (4.18) can be solved using equations (2.13), (2.14), reducing the nonlinearity in equation (4.16) to a quantity containing only space derivatives.

It is worth observing that equations (4.9) contain a trivial solution given by $\mathcal{R}_{kl,x} = I_{kl}$, as it follows by comparing equations (4.9) with equations (3.12). This particular solution permits us to define a set of generators σ_{kl} which eliminate completely the quantities $\mathcal{J}_{kl,x} + I_{kl}$ from the currents (3.15) and transform the system of CNSEs (2.10) into another one containing nonlocal nonlinearities. Such a situation, although interesting, is outwith the purpose of the present work.

In conclusion, let us describe briefly the generalization of the method to the case of noncanonical systems.

Firstly, we observe that for a noncanonical system the two matrices $\widehat{W}$ and $\widehat{\mathcal{W}}$ are not derivable from a potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$. In particular, the matrix $\widehat{W}$ can assume any arbitrary expression.

Differently, the form of the matrix $\widehat{\mathcal{W}}$ is constrained by the existence of the set of the continuity equations. Without loss of generality we can pose

$$2\rho_{kl}\mathcal{W}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] = -(\mathcal{J}_{kl,x}[\boldsymbol{\rho}, \boldsymbol{S}] + I_{kl}[\boldsymbol{\rho}, \boldsymbol{S}]), \tag{4.21}$$

where now the functionals $\mathcal{J}_{kl}$ and I_{kl} are no longer related to the nonlinear potential $U[\boldsymbol{\rho}, \boldsymbol{S}]$ through equations (3.8) and (3.10). The continuity equations (1.4) require that the functionals I_{kl} still fulfil the constraints (3.12) for an arbitrary set of functionals $G_k[\boldsymbol{\rho}, \boldsymbol{S}]$. The total currents J_k are given in equation (3.14) but now the functionals $\mathcal{J}_{kl}$ and G_{kl} are related to the matrix $\widehat{\mathcal{W}}$ only through equation (4.21).

At this point, by following the same steps described for the canonical case, it is easy to verify that the transformation (4.1) with generators (4.8) eliminates the anti-Hermitian matrix $\widehat{\mathcal{W}}$ of the nonlinearity, and transforms the system of CNSEs in the form given in equation (4.16) with only a Hermitian matrix $\widehat{W}'$ which is given again through equations (4.17)–(4.19).

5. Applications

In order to show the relevance of the method described in this paper, let us consider the following Hermitian and anti-Hermitian nonlinearities given by

$$\widehat{W}[\boldsymbol{\rho}, \boldsymbol{S}] = \text{diag}\left[\sum_{i=1}^{p} \rho_i(b_{ij}S_{j,x} + c_{ij}S_{i,x}) + f_j(\boldsymbol{\rho})\right], \tag{5.1}$$

$$\widehat{\mathcal{W}}[\boldsymbol{\rho}] = \text{diag}\left[\sum_{i=1}^{p}\left(d_{ij}\frac{\rho_i}{\rho_j}\rho_{j,x} + e_{ij}\rho_{i,x}\right)\right], \tag{5.2}$$

where b_{ij}, c_{ij}, d_{ij} and e_{ij} are real constants and $f_j(\boldsymbol{\rho})$ are arbitrary real functionals depending only on the vector field $\boldsymbol{\rho}$.

For the sake of simplicity, in this example we recover the more easy notation $\psi_{kl} \to \psi_j$ with $j = 1, \ldots, p$ and deal with the only two particular cases (a) and (b) discussed in the introduction.

The system of CNSEs (1.1), with the two nonlinearities (5.1) and (5.2), is given by

$$\mathrm{i}\psi_{j,t} = -a_j \psi_{j,xx} + f_j(\boldsymbol{\rho})\psi_j + v_j(x)\psi_j + \mathrm{i}\sum_{i=1}^{p}\left(\alpha_{ij}\frac{\rho_i}{\rho_j}\psi_j\psi_{j,x}^*\psi_j + \beta_{ij}\rho_i\psi_{j,x} + \gamma_{ij}\psi_i\psi_{i,x}^*\psi_j + \epsilon_{ij}\psi_i^*\psi_{i,x}\psi_j\right), \tag{5.3}$$

with $\alpha_{ij} = d_{ij} + b_{ij}/2$, $\beta_{ij} = d_{ij} - b_{ij}/2$, $\gamma_{ij} = e_{ij} + c_{ij}/2$ and $\epsilon_{ij} = e_{ij} - c_{ij}/2$. Equation (5.3), with $v_j(x) = 0$, includes some cases already known in the literature. For instance: the vector generalization of the Kaup–Newell equation [32] ($a_j = 1, c_{ij} = 0, -b_{ij} = 2d_{ij} = e_{ij} = \beta$ and $f_j(\boldsymbol{\rho}) = 0$); the coupled Chen–Lie–Liu equation (type I) [33] ($a_j = 1, c_{ij} = e_{ij} = 0, -b_{ij} = 2d_{ij} = \beta, f_j(\boldsymbol{\rho}) = 0$); the coupled Chen–Lie–Liu equation (type II) [33] ($a_j = 1, b_{ij} = d_{ij} = 0, c_{ij} = -2e_{ij} = \beta, f_j(\boldsymbol{\rho}) = 0$); the hybrid CNSE [34, 35] ($a_j = 1, c_{ij} = 0, -b_{ij} = 2d_{ij} = e_{ij} = \beta$ and $f_j(\boldsymbol{\rho}) = \beta\sum_k \rho_k$); the vectorial Eckhaus equation [36] ($\alpha_{ij} = 0, f_j(\boldsymbol{\rho}) = \sum_{ik}\lambda_{jik}\rho_i\rho_k$). Moreover, for $q = p = 2$, with $b_{ij} + 2d_{ij} = 0$ and $f_1(\boldsymbol{\rho}) = f\rho_1 + g\rho_2$, $f_2(\boldsymbol{\rho}) = g\rho_1 + f\rho_2$, equation (5.3) has been studied in [37].

The canonical sub-family of equation (5.3) is given by posing $b_{ij} = c_{ji} = -2d_{ij} = -2e_{ij}$, with the nonlinear potential

$$U[\boldsymbol{\rho}, \boldsymbol{S}] = -\sum_{i,j=1}^{p} b_{ij}\rho_i\rho_j S_{i,x} + F(\boldsymbol{\rho}), \tag{5.4}$$

where the conditions $\delta F(\boldsymbol{\rho})/\delta\rho_j = f_j(\boldsymbol{\rho})$ are assumed.

It is easy to observe that:

(a) when $d_{ij} = e_{ij}$, for $i \neq j$, equation (5.3) conserves the densities $\rho_j = |\psi_j|^2$, and the currents take the form

$$J_j = 2a_j\rho_j S_{j,x} - (d_{jj} + e_{jj})\rho_j^2 - 2\sum_{i=1, i\neq j}^{p} d_{ij}\rho_i\rho_j, \tag{5.5}$$

with $\mathcal{J}_j(\boldsymbol{\rho}) = -(d_{jj} + e_{jj})\rho_j^2 - 2\sum_{i\neq j} d_{ij}\rho_i\rho_j$ and $I_j(\boldsymbol{\rho}) = 0$.

(b) when $d_{ij} + e_{ji} = d_{ji} + e_{ij}$, equation (5.3) conserves the total density $\rho = \sum_j \rho_j$, and the total current is given by

$$J = \sum_{j=1}^{p}\left[2a_j\rho_j S_{j,x} - \sum_{i=1}^{p}(d_{ij} + e_{ji})\rho_i\rho_j\right], \tag{5.6}$$

with $\mathcal{J}_j(\boldsymbol{\rho}) = -(d_{jj} + e_{jj})\rho_j^2$ and $I_j(\boldsymbol{\rho}) = -2\sum_{i\neq j}\left(d_{ij}\rho_i\rho_{j,x} + e_{ij}\rho_j\rho_{i,x}\right)$.

If we choose the functionals $\mathcal{R}_j(\boldsymbol{\rho}) = 0$ in the case (a) and

$$\mathcal{R}_j(\boldsymbol{\rho}) = -\sum_{i=1, i\neq j}^{p}\lambda_{ij}\rho_i\rho_j, \tag{5.7}$$

in the case (b), where $\lambda_{ij} = d_{ij} + e_{ij}$, the generators (4.8) can be written in the unified form

$$\sigma_j(\boldsymbol{\rho}) = -\frac{1}{2a_j}\sum_{i=1}^{p}\lambda_{ij}\int \rho_i\,\mathrm{d}x. \tag{5.8}$$

By performing transformation (4.1) in equation (5.3) we obtain a new system of CNSEs for the field Φ containing the nonlinearity

$$\widehat{W}'[\boldsymbol{\rho}, \boldsymbol{S}] = \widehat{D}[\boldsymbol{\rho}, \boldsymbol{S}] + \widehat{C}[\boldsymbol{\rho}, \boldsymbol{S}], \tag{5.9}$$

where the diagonal matrix $\widehat{D}$ has entries

$$\widehat{D}[\boldsymbol{\rho}, \boldsymbol{S}] = \mathrm{diag}\left[\sum_{i=1}^{p} \rho_i(\mu_{ij}\Sigma_{j,x} + \nu_{ij}\Sigma_{i,x}) + \sum_{i,k=1}^{p} \omega_{jik}\rho_i\rho_k + f_j(\boldsymbol{\rho})\right], \tag{5.10}$$

with

$$\begin{aligned} \mu_{ij} &= b_{ij} + \lambda_{ij}, \\ \nu_{ij} &= c_{ij} - \frac{a_i}{a_j}\lambda_{ij}, \\ \omega_{jik} &= \frac{1}{4a_j}\left(\lambda_{ij}\lambda_{kj} + 2b_{ij}\lambda_{kj} + 2\frac{a_j}{a_i}c_{ij}\lambda_{ki}\right), \end{aligned} \tag{5.11}$$

whereas the off-diagonal matrix $\widehat{C}$ has entries

$$(\widehat{C}[\boldsymbol{\rho}, \boldsymbol{S}])_{ij} = \mathrm{i}\frac{\mathcal{F}_i(\boldsymbol{\rho}) - \mathcal{F}_j(\boldsymbol{\rho})}{2p\sqrt{\rho_i\rho_j}}\,\mathrm{e}^{\mathrm{i}(\Sigma_i - \Sigma_j)}, \tag{5.12}$$

with

$$\mathcal{F}_j(\boldsymbol{\rho}) = \sum_{i=1}^{p}(d_{ij} - e_{ij})(\rho_i\rho_{j,x} - \rho_{i,x}\rho_j). \tag{5.13}$$

We observe that the functionals (5.13) vanish in the case (a) and the nonlinearity $\widehat{W}'$ reduces to a purely real one.

Depending on the initial parameters b_{ij}, c_{ij}, d_{ij}, e_{ij} and on the functionals $f_j(\boldsymbol{\rho})$, the new CNSEs contain some interesting cases.

For instance, by choosing $b_{ij} = -2d_{ij} = -2e_{ij}$ and $a_jc_{ij} = 2a_id_{ij}$ we obtain a system of CNSEs with a purely real nonlinearity, which depends only on the fields ρ_i

$$\mathrm{i}\phi_{j,t} = -a_j\phi_{j,xx} + \left(\sum_{i,k=1}^{p} \omega_{jik}\rho_j\rho_k + f_j(\boldsymbol{\rho})\right)\psi_j + v_j(x)\phi_j. \tag{5.14}$$

In particular, the vectorial Eckhaus equation, with $\lambda_{jik} = \sum_{ik} b_{ij}(b_{kj} - 2b_{ki})/4a_j$, is reduced to a system of decoupled linear Schrödinger equations

$$\mathrm{i}\phi_{j,t} = -a_j\phi_{j,xx}. \tag{5.15}$$

Other interesting cases can be found by inspection [28].

6. Final comments

In this paper we have generalized a method, previously presented in the literature, for the $U(1)$-invariant nonlinear Schrödinger equations with a complex nonlinearity to the case of $U(1)$-invariant system of coupled nonlinear Schrödinger equations containing a very general nonlinearity $\widehat{\Lambda}[\boldsymbol{\rho}, \boldsymbol{S}]$. Without loss of generality, such nonlinear system can be arranged in a diagonal form where the nonlinearity $\widehat{W}[\boldsymbol{\rho}, \boldsymbol{S}] + \mathrm{i}\widehat{\mathcal{W}}[\boldsymbol{\rho}, \boldsymbol{S}]$ is given by two diagonal matrices with real entries. For such a system, we have introduced a nonlinear and unitary transformation changing the initial nonlinearity in another purely Hermitian. Consequently, the nonlinear currents (3.14) associated with the conserved densities $\rho_k = \sum_l \rho_{kl}$ are transformed into the standard bilinear currents (1.7). Moreover, it has been shown that when the system conserves separately all the quantities N_{k1}, the Hermitian matrix $\widehat{W}'$ contains only purely real entries. Extension of the method to noncanonical systems has been discussed.

In particular, we have shown that, starting from a given set of $U(1)$-invariant coupled nonlinear Schrödinger equations, there are many different possibilities of defining the generators of the transformation, as given in equations (4.8) and (4.9). For any of these choices we obtain a new set of coupled nonlinear evolution equations with a different, but Hermitian, nonlinearity, through equations (4.18) and (4.19). The transformed systems, in spite of their different nonlinearities, are all physically equivalent because, due to the unitarity of the gauge transformation, the conserved density fields ρ_k are equal in space at all time.

Generalization to high spatial dimensions is also immediate. In this case, according to equations (4.8), the generators of the transformation $\sigma_{kl}[\boldsymbol{\rho}, \boldsymbol{S}]$ can be introduced through the relations

$$\nabla\sigma_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] = \frac{\boldsymbol{\mathcal{J}}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] + \boldsymbol{\mathcal{R}}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}]}{2a_{kl}\rho_{kl}}, \tag{6.1}$$

where

$$\boldsymbol{\mathcal{J}}_{kl} = 2a_{kl}\rho_{kl}\nabla S_{kl}, \tag{6.2}$$

and $\boldsymbol{\mathcal{R}}_{kl}$ denotes the vectorial generalization of the functionals $\mathcal{R}_{kl}$ introduced through equations (3.12) and (4.9). We remark that the following conditions of consistence must be fulfilled [22]

$$\nabla \times \left[\frac{1}{\rho_{kl}}(\boldsymbol{\mathcal{J}}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}] + \boldsymbol{\mathcal{R}}_{kl}[\boldsymbol{\rho}, \boldsymbol{S}])\right] = 0. \tag{6.3}$$

Equations (6.3) select the potentials $U[\boldsymbol{\rho}, \boldsymbol{S}]$ and, through equations (2.11), (2.12), the nonlinear system, where the transformation can be performed. For noncanonical systems, equations (6.3) constraint only the form of the anti-Hermitian matrix $\widehat{\mathcal{W}}$, as can be seen through equation (4.21).

Let us observe that the transformation (4.1) typically breaks the canonical structure of the theory. As a consequence, the new system of CNSEs does not admit, in general, a Lagrangian formulation. The opposite is also true. When the transformation is applied to a noncanonical system, the new system of CNSEs may acquire a canonical structure. A sufficient condition to obtain a canonical system, after transformation, is given by observing that, if the transformed nonlinear matrix $\widehat{W}'$ is a functional depending only on the field ρ, from equation (2.11) it follows that also the nonlinear potential U depends only on ρ and consequently the anti-Hermitian matrix $\widehat{\mathcal{W}}'$, given by equation (2.12), vanishes.

In conclusion, we have presented some examples to show the applicability of the method. Although some of these are known before in the literature, it has been shown that the nonlinear transformation introduced in this paper allows us to treat in a unifying scheme all these CNSEs, obtaining, in a systematic way, the transformations introduced by the different authors.

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